Generating entanglement of photon-number states with coherent light via cross-Kerr nonlinearity

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Abstract

We propose a scheme for generating entangled states of light fields. This scheme only requires the cross-Kerr nonlinear interaction between coherent light-beams, followed by a homodyne detection. Therefore, this scheme is within the reach of current technology. We study in detail the generation of the entangled states between two modes, and that among three modes. In addition to the Bell states between two modes and the W states among three modes, we find plentiful new kinds of entangled states. Finally, the scheme can be extend to generate the entangled states among more than three modes.

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1 Introduction

Entanglement is a characteristic feature of quantum states and has important applications in quantum science and technology, for example, in quantum computation and quantum information [1]. There are a lot of schemes for generating various kinds of entanglement, for example, the entanglement between photons, the entanglement between atoms, the entanglement between trapped ions, and the entanglement between different kinds of particles (for example, between photons and atoms). In addition to the entanglement between two parties, there are also entanglement of multiparties. Among these schemes many use single-photon sources and/or single-photon detectors. Although there are great progresses in the study on these single-photon devices, how to obtain them is still a challenging task. In this paper we propose a simple scheme for generating entangled states of light fields. This scheme only requires the cross-Kerr nonlinear interaction between light fields in coherent states, followed by a homodyne detection. Therefore, this scheme is within the reach of current technology.The basic idea of this scheme is shown in Figure 1. Mode a is a bright beam which is in a coherent state $|\alpha\rangle$. Mode b is a weak or bright beam which is also in a coherent state. BS is a 50/50 beam splitter. $KM1$ and $KM2$ are Kerr media. HD means homodyne detection [2].

This paper is organized as follows: In section 2 we briefly introduce the cross-Kerr nonlinear interaction between two field-modes. In section 3 and section 4 we study the generation of entanglement between two modes and that among three modes, respectively. Section 5 is a summary.

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2 Cross-Kerr nonlinear interaction

First, let us briefly review the cross-Kerr nonlinear interaction between a mode A and a mode B. The interaction Hamiltonian has the form [3]

$$H_{CK} = \hbar K \hat{n}_A \hat{n}_B, \tag{1}$$

where $\hat{n}_A$ and $\hat{n}_B$ are the photon-number operator of mode A and mode B, respectively. The coupling coefficient K is proportional to the third-order nonlinear susceptibility $\chi^{(3)}$. The time-evolution operator is

$$U(t) = \exp\left(-\frac{i}{\hbar} H_{CK} t\right) = \exp\left\{-iK\hat{n}_A \hat{n}_B t\right\} = \exp\left\{-i\tau \hat{n}_A \hat{n}_B\right\} = U(\tau), \tag{2}$$

in which $\tau = Kt = K(l/v)$, it can be named as the scaled interaction time, or the nonlinear phase shift. Here l is the length of the Kerr medium and v is the velocity of light in the Kerr medium. The cross Kerr nonlinearity has following property

$$U(\tau)\left|n\right\rangle_B \left|\alpha\right\rangle_A = \left|n\right\rangle_B \left|\alpha e^{-in\tau}\right\rangle_A, \tag{3}$$

here $|n\rangle$ and $|\alpha\rangle$ are the photon number state and the coherent state, respectively.

3 Entanglement between two modes

Now let us study the generation of the entangled states between two modes. The scheme is shown in Figure 1. Assume that mode a is in a coherent state $|\alpha\rangle$ [4]. Mode b is also in a coherent state which is divided by the 50/50 beam splitter BS into two beams $b1$ and $b2$, and both $b1$ and $b2$ are in coherent state $|\beta\rangle$.

We first consider the case of weak coherent state $|\beta\rangle$. In this case we have

$$|\beta\rangle \approx \frac{1}{\sqrt{1+|\beta|^2}}\left(|0\rangle + \beta|1\rangle\right), \tag{4}$$

where $|0\rangle$ and $|1\rangle$ are the vacuum state and one-photon state, respectively. Let mode a interacts with mode $b1$ and $b2$ successively. For simplicity, we assume that both the scaled interaction times are τ, thai is, $\tau_1 = K_1 t_1 = \tau_2 = K_2 t_2 = \tau$.The interactions change the state as following way

$$|\beta\rangle_2 |\beta\rangle_1 |\alpha\rangle_a \rightarrow \frac{1}{1+|\beta|^2}\left[|0\rangle_2 |0\rangle_1 |\alpha\rangle_a + \beta\left(|1\rangle_2 |0\rangle_1 + |0\rangle_2 |1\rangle_1\right)\left|\alpha e^{-i\tau}\right\rangle_a + \beta^2 |1\rangle_2 |1\rangle_1 \left|\alpha e^{-i2\tau}\right\rangle_a\right], \tag{5}$$

where the subscripts 1 and 2 denote modes $b1$ and $b2$, respectively. We note that the internal product of coherent states satisfies [4]

$$\left|\left\langle \alpha e^{-in\tau} | \alpha e^{-i(n+1)\tau}\right\rangle\right|^2 = e^{-4|\alpha|^2 \sin^2(\tau/2)} \approx e^{-|\alpha|^2\tau^2}, \tag{6}$$

in which we have taken into account the fact that in practice τ is small [3] and therefore $\sin(\tau/2) \approx \tau/2$. However, if mode a is bright enough so that $|\alpha|^2 \tau^2 \gg 1$, then the two coherent states will be approximately orthogonal. This condition can be easily satisfied in experiments and in following discussions we assume that it is satisfied. In this case, a homodyne detection can distinguish different

coherent states [5]. Therefore, when we find that mode a is in the coherent state $\left|\alpha e^{-i\tau}\right\rangle_a$, then beam $b1$ and beam $b2$ will be projected into the entangled state

$$\frac{1}{\sqrt{2}}\left(|1\rangle_2|0\rangle_1+|0\rangle_2|1\rangle_1\right), \tag{7}$$

and the probability for getting this entangled state is $2|\beta|^2/\left(1+|\beta|^2\right)^2$ .This state is one of Bell states [1] and a special case of the $NOON$ states [6].

Now let us consider the general situation in which beam $b1$ and beam $b2$ are normal coherent states [4]. In this situation,

$$|\beta\rangle=\exp\left(-\frac{1}{2}|\beta|^2\right)\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle . \tag{8}$$

The cross-Kerr interactions transform the state as follows

$$\begin{aligned}|\beta\rangle_2|\beta\rangle_1|\alpha\rangle_a &= e^{-|\beta|^2}\sum_{m,n}\frac{\beta^{m+n}}{\sqrt{m!n!}}|m\rangle_2|n\rangle_1|\alpha\rangle_a\\ &\rightarrow e^{-|\beta|^2}\sum_{m,n}\frac{\beta^{m+n}}{\sqrt{m!n!}}|m\rangle_2|n\rangle_1\left|\alpha e^{-i(m+n)\tau}\right\rangle_a .\end{aligned} \tag{9}$$

If the homedyne detection finds mode a in the state $\left|\alpha e^{-i(m+n)\tau}\right\rangle_a=\left|\alpha e^{-ik\tau}\right\rangle_a$ ($k=m+n=1,2,...$), then mode $b1$ and mode $b2$ will be collapse into the entangled state

$$\frac{1}{\sqrt{2^k}}\sum_{n=0}^{k}\sqrt{\frac{k!}{n!\,(k-n)!}}|k-n\rangle_2|n\rangle_1 \qquad (k=1,2,...). \tag{10}$$

Since in this state the sum of photon numbers of the two modes is equal to k, we name this state as the 2-mode k-photon entangled state. The probability for getting this state is $exp(-2|\beta|^2)\frac{2^k}{k!}|\beta|^{2k}$. The entanglement property of the states expressed by Eq.(10) can be proved by using following entanglement criteria [7]

$$\left|\left\langle b_1^+b_2\right\rangle\right|^2>\left\langle N_{b1}N_{b2}\right\rangle, \tag{11}$$

where $N_{b1}(N_{b2}), b_1(b_2)$ and $b_1^+(b_2^+)$ are the photon-number operator, the photon annihilation operator and the photon creation operator of mode $b1(b2)$, respectively. For the states of equation (10), we can find $\left|\left\langle b_1^+b_2\right\rangle\right|^2=\frac{1}{4}k^2$, and $\left\langle N_{b1}N_{b2}\right\rangle=\frac{1}{4}k\,(k-1)$. Therefore the entanglement condition (11) is satisfied, and the states (10) are indeed entangled states. For $k=1$, equation (10) reduces to equation (7), and some other examples of the 2-mode k-photon entangled states are listed below.

$$\frac{1}{2}\left[\left(|2\rangle_2|0\rangle_1+|0\rangle_2|2\rangle_1\right)+\sqrt{2}|1\rangle_2|1\rangle_1\right] \qquad (k=2) \tag{12}$$

$$\frac{1}{\sqrt{8}}\left[\left(|3\rangle_2|0\rangle_1+|0\rangle_2|3\rangle_1\right)+\sqrt{3}\left(|2\rangle_2|1\rangle_1+|1\rangle_2|2\rangle_1\right)\right] \qquad (k=3) \tag{13}$$

Equations (12) and (13) are new kinds of entangled states. Equation (12) can be understood as a superposition of a $NOON$ state $\left(|2\rangle_2|0\rangle_1+|0\rangle_2|2\rangle_1\right)$ and a product state $|1\rangle_2|1\rangle_1$,while equation (13) can be understood as a superposition of a $NOON$ state $\left(|3\rangle_2|0\rangle_1+|0\rangle_2|3\rangle_1\right)$ and a $NOON-like$ state $\left(|2\rangle_2|1\rangle_1+|1\rangle_2|2\rangle_1\right)$. We also note that in the superposition (13) the probability of getting the state $\left(|2\rangle_2|1\rangle_1+|1\rangle_2|2\rangle_1\right)$ is larger than that of getting the state $\left(|3\rangle_2|0\rangle_1+|0\rangle_2|3\rangle_1\right)$.That is, the photons trend to distribute between the two modes symmetrically. The properties and applications of these new kinds of entangled states will be studied in the future.

4 Entanglement among three modes

We can extend the scheme above to generate the entanglement among three modes. For this purpose we modify the scheme from Figure 1 to Figure 2, in which BS1 has the $reflection/transmission = 1/2$ and BS2 has the $reflection/transmission = 1/1$, so that the three beams $b1, b2$ and $b3$ have the same strength, and we assume all of them are in the coherent state $|\beta\rangle$. We let mode a, in a coherent state $|\alpha\rangle$, interacts with modes $b1, b2$ and $b3$ successively. And for simplicity, we assume that all of the scaled interaction times are equal, thai is, $\tau_1 = \tau_2 = \tau_3 = \tau$.

For the situation in which $|\beta\rangle$ is weak and can be expressed as in equation (4), the interactions transform the states in the following way

$$\begin{aligned} |\beta\rangle_3 |\beta\rangle_2 |\beta\rangle_1 |\alpha\rangle_a \quad \rightarrow \quad & \frac{1}{\left(1+|\beta|^2\right)^{3/2}} \{ |0\rangle_3 |0\rangle_2 |0\rangle_1 |\alpha\rangle_a \\ & + \beta \left(|1\rangle_3 |0\rangle_2 |0\rangle_1 + |0\rangle_3 |1\rangle_2 |0\rangle_1 + |0\rangle_3 |0\rangle_2 |1\rangle_1 \right) \left| \alpha e^{-i\tau} \right\rangle_a \\ & + \beta^2 \left(|1\rangle_3 |1\rangle_2 |0\rangle_1 + |1\rangle_3 |0\rangle_2 |1\rangle_1 + |0\rangle_3 |1\rangle_2 |1\rangle_1 \right) \left| \alpha e^{-i2\tau} \right\rangle_a \\ & + \beta^3 |1\rangle_3 |1\rangle_2 |1\rangle_1 \left| \alpha e^{-i3\tau} \right\rangle_a \}. \end{aligned} \tag{14}$$

As discussed above, we assume that different coherent states in above equation are approximately orthogonal, and we can use homodyne detection to distinguish them [5]. If we find that mode a is in state $\left|\alpha e^{-i\tau}\right\rangle_a$ then modes $b1, b2$ and $b3$ will be projected to the entangled state

$$\frac{1}{\sqrt{3}} \left(|1\rangle_3 |0\rangle_2 |0\rangle_1 + |0\rangle_3 |1\rangle_2 |0\rangle_1 + |0\rangle_3 |0\rangle_2 |1\rangle_1 \right), \tag{15}$$

and the probability for obtaining this state is $3|\beta|^2 / \left(1+|\beta|^2\right)^3$. On the other hand, If we find that mode a is in state $\left|\alpha e^{-i2\tau}\right\rangle_a$ then modes $b1, b2$ and $b3$ will be projected to the entangled state

$$\frac{1}{\sqrt{3}} \left(|1\rangle_3 |1\rangle_2 |0\rangle_1 + |1\rangle_3 |0\rangle_2 |1\rangle_1 + |0\rangle_3 |1\rangle_2 |1\rangle_1 \right), \tag{16}$$

and the probability for getting this state is $3|\beta|^4 / \left(1+|\beta|^2\right)^3$. Equations (15) and (16) can be named as 1-photon W state [8] and 2-photon W state, respectively.

For the general case in which $|\beta\rangle$ is not very weak we use equation (8). In this case the interactions transform the states as follows:

$$\begin{aligned} |\beta\rangle_3 |\beta\rangle_2 |\beta\rangle_1 |\alpha\rangle_a \quad &= \quad e^{-3|\beta|^2/2} \sum_{l,m,n} \frac{\beta^{l+m+n}}{\sqrt{l!m!n!}} |l\rangle_3 |m\rangle_2 |n\rangle_1 |\alpha\rangle_a \\ &\rightarrow \quad e^{-3|\beta|^2/2} \sum_{l,m,n} \frac{\beta^{l+m+n}}{\sqrt{l!m!n!}} |l\rangle_3 |m\rangle_2 |n\rangle_1 \left| \alpha e^{-i(l+m+n)\tau} \right\rangle_a . \end{aligned} \tag{17}$$

If we find that mode a is in the state $\left|\alpha e^{-i(l+m+n)\tau}\right\rangle_a = \left|\alpha e^{-ik\tau}\right\rangle_a$ $(k = l+m+n = 1, 2, ...)$, then modes $b1, b2$ and $b3$ will be projected to the entangled state

$$\frac{1}{\sqrt{3^k}} \sum_{m=0}^{k} \sum_{n=0}^{k-m} \sqrt{\frac{k!}{(k-m-n)!m!n!}} |k-m-n\rangle_3 |m\rangle_2 |n\rangle_1 \qquad (k = 1, 2, ...). \tag{18}$$

We name this state as the 3-mode k-photon entangled state. The probability for getting this state is $exp(-3|\beta|^2)\frac{3^k}{k!}|\beta|^{2k}$ .The entanglement property of the states of Eq.(18) can be proved by using following entanglement criteria [7]

$$\left|\left\langle b_1^+ b_2\right\rangle\right|^2 > \left\langle N_{b1} N_{b2}\right\rangle \quad \text{and} \quad \left|\left\langle b_2^+ b_3\right\rangle\right|^2 > \left\langle N_{b2} N_{b3}\right\rangle . \tag{19}$$

For the states (18), we can find $\left|\left\langle b_1^+ b_2\right\rangle\right|^2 = \left|\left\langle b_2^+ b_3\right\rangle\right|^2 = \frac{1}{9}k^2$, and $\left\langle N_{b1} N_{b2}\right\rangle = \left\langle N_{b2} N_{b3}\right\rangle = \frac{1}{9}k\,(k-1)$ .Therefore the entanglement condition (19) is satisfied, and the states (18) are indeed entangled states of three modes. For $k = 1$, equation (18) reduces to equation (15), and some other examples of the 3-mode k-photon entangled state are as follows:

$$\begin{aligned}&\frac{1}{3}\{(\left|2\right\rangle_3 \left|0\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|2\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|0\right\rangle_2 \left|2\right\rangle_1)\\ &+\sqrt{2}\,(\left|1\right\rangle_3 \left|1\right\rangle_2 \left|0\right\rangle_1 + \left|1\right\rangle_3 \left|0\right\rangle_2 \left|1\right\rangle_1 + \left|0\right\rangle_3 \left|1\right\rangle_2 \left|1\right\rangle_1)\} \qquad (k=2)\,,\end{aligned} \tag{20}$$

$$\begin{aligned}&\frac{1}{\sqrt{3^3}}\{(\left|3\right\rangle_3 \left|0\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|3\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|0\right\rangle_2 \left|3\right\rangle_1)\\ &+\sqrt{3}\,(\left|2\right\rangle_3 \left|1\right\rangle_2 \left|0\right\rangle_1 + \left|2\right\rangle_3 \left|0\right\rangle_2 \left|1\right\rangle_1 + \left|1\right\rangle_3 \left|2\right\rangle_2 \left|0\right\rangle_1 + \left|1\right\rangle_3 \left|0\right\rangle_2 \left|2\right\rangle_1 + \left|0\right\rangle_3 \left|2\right\rangle_2 \left|1\right\rangle_1 + \left|0\right\rangle_3 \left|1\right\rangle_2 \left|2\right\rangle_1)\\ &+\sqrt{6}\left|1\right\rangle_3 \left|1\right\rangle_2 \left|1\right\rangle_1\} \qquad (k=3)\,.\end{aligned} \tag{21}$$

Equation (20) is a superposition of two 2-photon W states. While equation (21) is a superposition of a 3-photon W state (the first line), a product state (the third line), and a state (the second line) which can be expressed as

$$\left|2\right\rangle_i \left(\left|1\right\rangle_j \left|0\right\rangle_k + \left|0\right\rangle_j \left|1\right\rangle_k\right) + \left|1\right\rangle_i \left(\left|2\right\rangle_j \left|0\right\rangle_k + \left|0\right\rangle_j \left|2\right\rangle_k\right) + \left|0\right\rangle_i \left(\left|2\right\rangle_j \left|1\right\rangle_k + \left|1\right\rangle_j \left|2\right\rangle_k\right), \tag{22}$$

where the subscripts $i = 1$,or 2, or 3,and j,k are the other two, respectively. We also note that in the superposition (20) the probability of getting the state $(\left|1\right\rangle_3 \left|1\right\rangle_2 \left|0\right\rangle_1 + \left|1\right\rangle_3 \left|0\right\rangle_2 \left|1\right\rangle_1 + \left|0\right\rangle_3 \left|1\right\rangle_2 \left|1\right\rangle_1)$ is larger than that of getting the state $(\left|2\right\rangle_3 \left|0\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|2\right\rangle_2 \left|0\right\rangle_1 + \left|0\right\rangle_3 \left|0\right\rangle_2 \left|2\right\rangle_1)$. This shows again that the photons trend to distribute among different modes symmetrically.

5 Summary

In summary, we have proposed a scheme for generating entangled states of light fields. This scheme has following advantages: First, the scheme only involves the cross-Kerr nonlinear interaction between coherent light-beams, followed by a homodyne detection. It is not necessary that the cross-Kerr nonlinearity is very large, as long as the coherent light is bright enough. Therefore, this scheme is within the reach of current technology. Second, in addition to the Bell states between two modes and the W states among three modes, plentiful new kinds of entangled states can be generated with this scheme. We also found that in the generated entangled states, the photons have a trend to distribute among different modes symmetrically. Finally, we would like to point out that the scheme can be extend to generate the entangled states among more than three modes.

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References

[1] N.A.Nielsen and I.L.Chuang,*Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press, 2000); D.Bouwmeester, A.Ekert, and A.Zeilinger, *The Physics of Quantum Information* (Berlin: Springer, 2000).

[2] U.Leonhardt,*Measuring the Quantum state of light* (Cambridge: Cambridge University Press,1997); H.A.Bachor and T.C.Ralph,*A Guide to Experiments in Quantum Optics*(Weinheim: Wiley-VCH Verlag GmbH & Co.KGaA, 2004).

[3] B.C.Sanders and G.J.Milburn, Phys. Rev. A **45**, 1919 (1992); C.C.Gerry, Phys. Rev. A **59,** 4095 (1999).

[4] C.C.Gerry and P.L.Knight, *Introductory Quantum Optics* (Cambridge: Cambridge University Press, 2005).

[5] K.Nemoto and W.J.Munro,Phys. Rev. Lett. 93, 250502 (2004); W.J.Munro,K.Nemoto, R.G.Beauoleil, and T.P.Spiller,Phys. Rev. A 71, 033819 (2005).

[6] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A 54, R4649 (1996); Z. Y. Ou, Phys. Rev. A 55, 2598 (1997); F.Shafiei, P.Srinivasan, and Z. Y. Ou,Phys. Rev. A 70, 043803 (2004); B.Lin and Z.Y.Ou, Phys. Rev. A 74, 035802 (2006); F.W.Sun, Z.Y.Ou, and G.C.Guo, Phys. Rev. A 73, 023808 (2006).

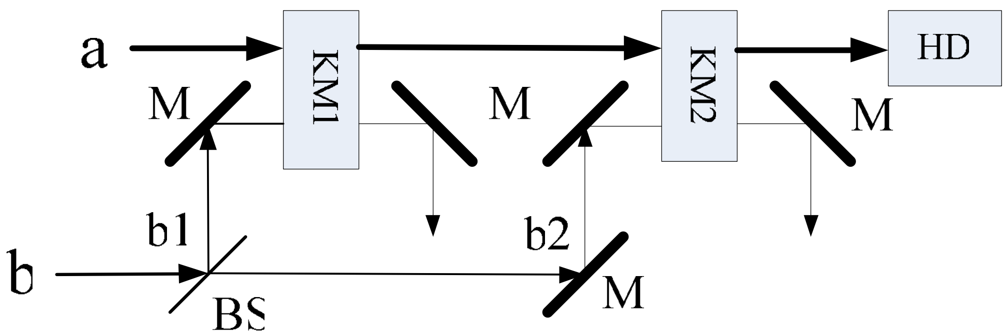
[7] M.Hillery and M.S.Zubairy, Phys. Rev. Lett. 96, 050503 (2006); M.Hillery and M.S.Zubairy, Phys. Rev. A 74, 032333 (2006).

[8] W.Dur, G.Vidal, and J.I.Cirac, Phys. Rev. A 62, 062314 (2000); P.Agrawal and A.Pati, Phys. Rev. A 74, 062320 (2006); R.Lohmayer, A.Osterloh, J.Siewert, and A.Uhlmann, Phys. Rev. Lett. 97, 260502 (2006); M.Bourennane, M.Eibl, S.Gaertner, N.Kiesel, C.Kurtsiefer, and H.Weinfurter, Phys. Rev. Lett. 96,100502 (2006); R.Rahimi, A.SaiToh, M.Nakahara, and M.Kitagawa, Phys. Rev. A 75, 032317 (2007).

Figure captions

Figure 1. Scheme for generating entanglement between two modes. KM:cross-Kerr medium; BS: beam splitter; M: mirror, HD: homodyne detection.

Figure 2. Scheme for generating entanglement among three modes. KM:cross-Kerr medium; BS: beam splitter; M: mirror, HD: homodyne detection.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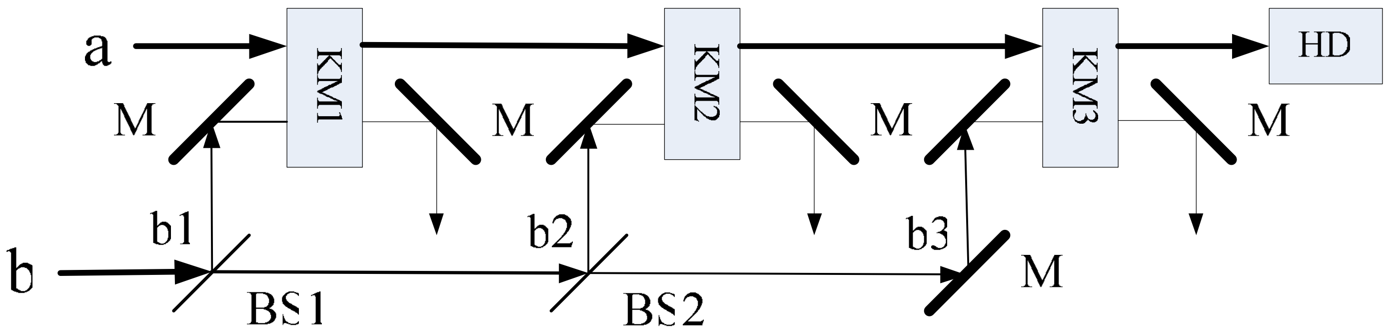
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